Slow inviscid flows of a compressible fluid in spatially inhomogeneous systems

V. P. Ruban*
Optics and Fluid Dynamics Department, Risø National Laboratory, DK-4000 Roskilde, Denmark
and L. D. Landau Institute for Theoretical Physics, 2 Kosygin Street, 117334 Moscow, Russia
(Received 24 October 2000; published 30 August 2001)

An ideal compressible fluid is considered, with an equilibrium density being a given function of coordinates due to presence of some static external forces. The slow flows in such system, which do not disturb the density, are investigated with the help of the Hamiltonian formalism. The equations of motion of the system are derived for an arbitrary given topology of the vorticity field. The general form of the Lagrangian for frozen-in vortex lines is established. The local induction approximation for motion of slender vortex filaments in several inhomogeneous physical models is studied.

DOI: 10.1103/PhysRevE.64.036305 PACS number(s): 47.15.Ki, 47.32.Cc, 47.37.+q, 47.65.+a

I. INTRODUCTION

Hydrodynamic-type systems of equations are extensively employed for macroscopic description of physical phenomena in ordinary and superfluid liquids, gases, plasmas, and in other substances. In solving hydrodynamic problems, it is admissible in many cases to neglect all dissipative processes and use the ideal fluid approximation, at least as the first step. With this approximation, a dynamic model describing flow is conservative. The Hamiltonian formalism is a convenient tool to deal with such systems [1,2], which makes possible to consider in a universal way all nonlinear processes. A big number of works is devoted to application of the Hamiltonian method in hydrodynamics (see, for instance, the reviews [3,4] and references therein).

One of the most important questions, permitted for a universal consideration in the frame of canonical formalism, is the question about integrals of motion of a dynamic system. According to the theorem of Noether [1,2], each conservation law of a system is closely connected to a symmetry of the corresponding Lagrangian with respect to some one-parameter group of transformations of dynamical variables. It is well known that the conservation laws for the energy, momentum, and the angular momentum follow from the fundamental properties of the space and time, namely, from homogeneity of the time and from homogeneity and isotropy of the space. Due to these properties, shifts and rotations of a system do not change its Lagrangian. The characteristic feature of the hydrodynamic-type systems is that they possess, besides the indicated usual integrals of motion, also an infinite number of specific integrals of motion related to the freezing-in property of canonical vorticity [3–12]. The reason for this is a basic physical property of fluids, relabeling symmetry. For instance, in isentropic flows the circulation of the canonical momentum along any frozen-in closed contour is conserved. In usual nonrelativistic hydrodynamics, where the canonical momentum coincides with the velocity, the given statement is known as the theorem of Kelvin about conservation of the velocity circulation [13,14].

Existence of an infinite number of integrals of motion influences strongly dynamical and statistical properties of liquid systems. This is the reason why a clarification of structure of conservation laws is very important, as well as the search for such new parametrizations for dynamical variables, which take into account the integrals of motion more completely. In many cases, even when a dissipation is present but its level is low, it is still correct to speak about integrals of the corresponding conservative problem, because values of some of them are conserved with a high accuracy, especially on an initial stage of the evolution, while the system has not proceeded to a state where a role of dissipation is significant due to large gradients. Besides this, conservation laws in physical systems, as a rule, are associated with definite geometrical objects. Usage of these associations promotes understanding and vivid imagination of everything that happens. In hydrodynamic models, the frozen-in vortex lines are such geometrical objects, so the present work is devoted to the study of the motion of vortex lines in spatially inhomogeneous systems.

Hydrodynamic equations describe, in particular, interaction between "soft" degrees of freedom of a system—frozen-in vortices, and "hard" degrees of freedom—acoustic modes. The presence of soft degrees of freedom is explained by the fact that equilibrium states of the fluid are highly degenerated due to the relabeling symmetry. Thus, no potential energy corresponds to soft degrees of freedom, unlike the hard degrees of freedom. Due to the dominating effect of elastic potential energy, hard degrees of freedom behave, typically, like a set of weakly nonlinear oscillators. On the contrary, dynamics of soft degrees of freedom is not dominated by a potential energy, and usually it is highly nonlinear. In a limit of slow flows, when a typical velocity of vortex structure motion is small in comparison with the sound speed, a dynamic regime is possible, in which the hard degrees of freedom, corresponding to deviations of fluid density $\rho(\mathbf{r},t)$ from an equilibrium configuration $\rho_0(\mathbf{r})$, are excited weakly. Then, completely neglecting the sound, in the homogeneous case $\rho_0=\mathrm{const}$ one arrives at the models of incompressible fluid. For dynamics of vortices in incompressible perfect fluid the so called formalism of vortex lines has been developed recently [8–12], which takes into account the conservation of topology of the vorticity field [15,16]. Application of this formalism allows one to deal with a partially integrated system, where the topology is

*Electronic address: ruban@itp.ac.ru

fixed by the Cauchy invariant [13]. In the proposed description the frozen-in solenoidal vorticity field is considered as a continuous distribution of the elementary objects—vortex lines. Such formulation of inviscid hydrodynamics as the problem of vortex line motion has been very suitable for the study of localized vortex structures such as vortex filaments. Also, it seems to be an adequate approach to the problem of finite time singularity formation in solutions of hydrodynamic equations [17].

The goal of the present work is to extend the vortex line formalism to the case where equilibrium density $\rho_0(\mathbf{r})$ is a fixed nontrivial function of spatial coordinates due to a static influence of some external forces. Such situation takes place in many physically important models. For examples, it can be the gravitational force for a large mass of an isentropic gas, both in usual and in relativistic hydrodynamics, or it can be the condition of electrical neutrality for the electron fluid on a given background of ion distribution in the model of electron magnetohydrodynamics (EMHD). The theory developed can be also applied to tasks about long-scale dynamics of the quantized vortex filaments in a Bose-Einstein condensate placed into a trap of a sufficiently large size. The vortex line formalism seems to be a universal and adequate tool for investigation of slow inviscid flows in inhomogeneous systems. For instance, it makes possible, in a simple and standard way, to analyze qualitative behavior of vortices without detailed consideration of basic equations of motion for the fluid. Therefore, the proposed approach can have advantages over other methods when complicated systems will be studied.

As a concrete result, the local induction approximation (LIA) in vortex dynamics will be analyzed for several spatially inhomogeneous physical systems, namely for Eulerian compressible hydrodynamics in an external field, for EMHD, and for vortices in trapped Bose-Einstein condensates. A new equation of vortex filament motion will be derived, which takes into account the inhomogeneity of these systems, Eq. (33). As to relativistic hydrodynamics in a static gravitational field, the proposed method gives a more complicated LIA equation than Eq. (33), as it has been shown recently by the present author [18]. (See also the most recent paper [19] about dynamics of an ultrarelativistic fluid in the flat anisotropic cosmological models of expanding universe, where the formalism of vortex lines has been applied to systems with Hamiltonian functionals depending explicitly on the time variable, and the effect of nonstationary anisotropy of the space on vortex dynamics has been studied.)

This paper is organized as follows. A short review of Lagrangian formalism for fluid media is given in Sec. II. It provides a basis for development in Sec. III of the vortex line formalism for spatially inhomogeneous systems. Then in Sec. IV, the method developed is applied to derive approximate equations of motion for slender nonstretched vortex filaments in three above-mentioned physical models.

II. LAGRANGIAN FORMALISM FOR A FLUID

From the viewpoint of the Lagrangian formalism, the freezing-in property of the canonical vorticity is due to the special symmetry of the basic equations of ideal hydrodynamics [3–8,11,12]. As known, the entire Lagrangian description of a motion of some continuous medium can be given by the three-dimensional (3D) mapping $\mathbf{r}=\mathbf{x}(\mathbf{a},t)$, which indicates the space coordinates of each medium point labeled by a label $\mathbf{a}=(a_1,a_2,a_3)$, at an arbitrary moment in time t. The labeling $\mathbf{a}$ can be chosen in such a manner that the amount of matter in a small volume $d^3\mathbf{a}$ in the label space is simply equal to this volume. With neglecting all dissipative processes, a dynamic model describing flow is conservative, so the equations of motion for the mapping $\mathbf{x}(\mathbf{a},t)$ follow from a variational principle

$$\delta S=\delta\int \mathcal{L}\{\mathbf{x}(\mathbf{a},t),\dot{\mathbf{x}}(\mathbf{a},t)\}dt=0,$$

where the Lagrangian $\mathcal{L}$ is a functional of $\mathbf{x}(\mathbf{a},t)$, $\dot{\mathbf{x}}(\mathbf{a},t)$, and also spatial derivatives. A very important circumstance is related to the fluidity property of the media under consideration. The fluidity is manifested in the fact that the Lagrangian actually contains the dependence on $\mathbf{x}(\mathbf{a},t)$ and $\dot{\mathbf{x}}(\mathbf{a},t)$ only through two Eulerian characteristics of the flow, namely through the field of density $\rho(\mathbf{r},t)$ and the velocity field $\mathbf{v}(\mathbf{r},t)$, i.e., $\mathcal{L}=\mathcal{L}\{\rho,\mathbf{v}\}$, with

$$\rho(\mathbf{r},t)=\det\left\|\frac{\partial\mathbf{a}(\mathbf{r},t)}{\partial\mathbf{r}}\right\|,\qquad \mathbf{v}(\mathbf{r},t)=\dot{\mathbf{x}}(\mathbf{a},t)\big|_{\mathbf{a}=\mathbf{a}(\mathbf{r},t)}. \quad (1)$$

Here $\mathbf{a}(\mathbf{r},t)$ is the inverse mapping with respect to $\mathbf{x}(\mathbf{a},t)$.

A simple particular example is the Lagrangian of ordinary Eulerian isentropic hydrodynamics

$$\mathcal{L}_{Euler}=\int\left(\rho\frac{\mathbf{v}^2}{2}-\varepsilon(\rho)-\rho U(\mathbf{r})\right)d\mathbf{r}, \quad (2)$$

where $\varepsilon(\rho)$ is the internal energy density and $U(\mathbf{r})$ is the external force potential, for instance, the gravitational potential.

A less trivial example is the Lagrangian of relativistic isentropic hydrodynamics [18] in a curved space time with metric tensor $g_{ik}(t,\mathbf{r})$ $(i,k=0\ldots3,\alpha,\beta=1\ldots3)$,

$$\mathcal{L}_r=-\int\mathcal{E}\left(\frac{\rho}{\sqrt{-g}}\sqrt{g_{00}+2g_{0\alpha}v^{\alpha}+g_{\alpha\beta}v^{\alpha}v^{\beta}}\right)\sqrt{-g}\,d\mathbf{r}.$$

Here $g=\det\|g_{ik}\|$ is the determinant of the metric tensor, the expression in parenthesis is equal to the absolute value of the current four-vector $n^i=n(dx^i/ds)$ [20]. A dependence $\mathcal{E}(n)$ connects the relativistic density $\mathcal{E}$ of the fluid energy, measured in a locally co-moving reference frame, with n.

In plasma physics, the model of electron magnetohydrodynamics is useful. EMHD follows in the limit of slow flows from the Lagrangian of electron fluid

$$\mathcal{L}_e=\int\left(\rho\frac{\mathbf{v}^2}{2}+\frac{e}{mc}\rho(\mathbf{A}\cdot\mathbf{v})-\frac{(\mathrm{curl}\,\mathbf{A})^2}{8\pi}+\cdots\right)d\mathbf{r}, \quad (3)$$

where $\rho(\mathbf{r},t)$ is the density of electron fluid, e is the electric charge of electron, m is its mass, and c is the speed of light.

The vector potential $\mathbf{A}(\mathbf{r},t)$ of the electromagnetic field determines the magnetic field $\mathbf{B}(\mathbf{r},t)$ by the relation $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$. In this paper, we will not need an explicit form of other terms indicated by the dots.

The list of examples, of course, is not exhausted by three given models. All known hydrodynamic models without dissipation, where the conservation of fluid amount takes place, can be described in this way. So the theory developed here is quite universal and applicable in various branches of physics where vortex phenomena occur.

It follows from the definitions (1) that dynamics of the density $\rho(\mathbf{r},t)$ obeys the continuity equation in its standard form

$$\rho_t+\nabla(\rho\mathbf{v})=0. \tag{4}$$

The vanishing condition for variation of the action $S=\int\mathcal{L}\{\rho,\mathbf{v}\}dt$, when the mapping $\mathbf{x}(\mathbf{a},t)$ is varied by $\delta\mathbf{x}(\mathbf{a},t)$, can be expressed in Eulerian representation as follows (the generalized Euler equation [11])

$$(\partial_t+\mathbf{v}\cdot\nabla)\left(\frac{1}{\rho}\cdot\frac{\delta\mathcal{L}}{\delta\mathbf{v}}\right)=\nabla\left(\frac{\delta\mathcal{L}}{\delta\rho}\right)-\frac{1}{\rho}\left(\frac{\delta\mathcal{L}}{\delta v^{\alpha}}\right)\nabla v^{\alpha}. \tag{5}$$

This is merely the variational Euler-Lagrange equation

$$\frac{d}{dt}\,\frac{\delta\mathcal{L}}{\delta\dot{\mathbf{x}}(\mathbf{a})}=\frac{\delta\mathcal{L}}{\delta\mathbf{x}(\mathbf{a})}$$

for fluid particle dynamics. The equations (4) and (5) determine completely evolution of hydrodynamic system.

It was already mentioned that in all such systems an infinite number of conservation laws exists. The $\{\rho,\mathbf{v}\}$ dependence means that the Lagrangian $\mathcal{L}\{\mathbf{x}(\mathbf{a},t),\dot{\mathbf{x}}(\mathbf{a},t)\}$ admits the infinite-parametric symmetry group—it assumes the same value on any two mappings $\mathbf{x}_1(\mathbf{a},t)$ and $\mathbf{x}_2(\mathbf{a},t)$, if they differ one from another only by some relabeling of the labels with unit Jacobian,

$$\mathbf{x}_2(\mathbf{a},t)=\mathbf{x}_1(\mathbf{a}^*(\mathbf{a}),t), \qquad \det\|\partial\mathbf{a}^*/\partial\mathbf{a}\|=1. \tag{6}$$

Obviously, such mappings create the same density and velocity fields. According to Noether's theorem [1,2], every one-parameter subgroup of the relabeling group $\mathbf{a}^*(\mathbf{a})$ with unit Jacobian corresponds to an integral of motion. There are several classifications of these conservation laws. For instance, one can postulate that circulation of the canonical momentum $\mathbf{p}(\mathbf{r},t)$,

$$\mathbf{p}(\mathbf{r},t)\equiv\left.\frac{\delta\mathcal{L}}{\delta\dot{\mathbf{x}}(\mathbf{a},t)}\right|_{\mathbf{a}=\mathbf{a}(\mathbf{r},t)}=\frac{1}{\rho}\left(\frac{\delta\mathcal{L}\{\rho,\mathbf{v}\}}{\delta\mathbf{v}}\right), \tag{7}$$

along an arbitrary frozen-in closed contour $\gamma(t)$ does not depend on time (the generalized theorem of Kelvin):

$$\oint_{\gamma(t)}(\mathbf{p}\cdot d\mathbf{r})=\mathrm{const}.$$

We arrive at a different formulation, in terms of the so-called Cauchy invariant, when we consider the solenoidal field of the canonical vorticity $\mathbf{\Omega}(\mathbf{r},t)$,

$$\mathbf{\Omega}(\mathbf{r},t)=\mathrm{curl}\,\mathbf{p}(\mathbf{r},t). \tag{8}$$

It is easy to check that application of the curl operator to the equation (5) gives

$$\mathbf{\Omega}_t=\mathrm{curl}\;[\mathbf{v}\times\mathbf{\Omega}]. \tag{9}$$

The formal solution of this equation is

$$\mathbf{\Omega}(\mathbf{r},t)=\int\delta(\mathbf{r}-\mathbf{x}(\mathbf{a},t))(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{x}(\mathbf{a},t)d\mathbf{a}, \tag{10}$$

where the solenoidal independent of time field $\mathbf{\Omega}_0(\mathbf{a})$ is exactly the Cauchy invariant. The equation (10) displays that lines of initial solenoidal field $\mathbf{\Omega}_0(\mathbf{a})$ are deformed in the course of motion by the mapping $\mathbf{x}(\mathbf{a},t)$, keeping all the topological characteristics unchanged [15,16]. This feature of vortex line dynamics is called the freezing-in property.

III. HAMILTONIAN DYNAMICS OF VORTEX LINES

To continue, it is more convenient to reformulate the problem in terms of density and canonical momentum. Let the system be specified by some Hamilton functional $\mathcal{H}\{\rho,\mathbf{p}\}$

$$\mathcal{H}=\int\left(\frac{\delta\mathcal{L}}{\delta\mathbf{v}}\cdot\mathbf{v}\right)d\mathbf{r}-\mathcal{L}, \tag{11}$$

where the velocity $\mathbf{v}$ is expressed through the momentum $\mathbf{p}$ and through the density ρ with the help of Eq. (7). Let us note that the following equality takes place:

$$\mathbf{v}=\frac{1}{\rho}\left(\frac{\delta\mathcal{H}}{\delta\mathbf{p}}\right), \tag{12}$$

which is analogous to formula (7). The Hamiltonian (noncanonical [3]) equations of motion for the fields of density and momentum follow from Eqs. (4) and (5). Taking into account the equality (12), they have the form (for detailed derivation see [11])

$$\rho_t+\nabla\left(\frac{\delta\mathcal{H}}{\delta\mathbf{p}}\right)=0, \tag{13}$$

$$\mathbf{p}_t=\left[\left(\frac{\delta\mathcal{H}}{\delta\mathbf{p}}\right)\times\frac{\mathrm{curl}\,\mathbf{p}}{\rho}\right]-\nabla\left(\frac{\delta\mathcal{H}}{\delta\rho}\right). \tag{14}$$

It is assumed in this work that the Hamiltonian has a minimum at some configuration $\{\rho_0(\mathbf{r}),\mathbf{p}_0(\mathbf{r})\}$. For simplicity, we will consider only systems without gyroscopic effects, i.e., $\mathbf{p}_0(\mathbf{r})=\mathbf{0}$. Our purpose is to study slow flows near the equilibrium. In the regime under consideration, which corresponds formally to "prohibition" of excitation of the acoustic modes, the flow of fluid occurs in such a way that the density of each moving portion of fluid follows the given function $\rho_0(\mathbf{r})$. Therefore, the equation (13) gives the condition

$$\nabla\left(\frac{\delta\mathcal{H}}{\delta\mathbf{p}}\right)=0, \tag{15}$$

which means that after imposing the constraint $\rho=\rho_0(\mathbf{r})$, the Hamiltonian no longer depends on the potential component of the canonical momentum field; it depends now only on the solenoidal component, i.e., actually on the vorticity $\mathbf{\Omega}$. It should be also noted that it is sufficient to take into consideration only the quadratic part of the Hamiltonian, because the flow is supposed to be slow, so higher order terms, if any exist, may be neglected. Therefore, in further equations, $\mathcal{H}\{\mathbf{\Omega}\}$ is actually a quadratic functional of the vorticity field, though this fact will not be used in formal calculations. The condition (15) implies validity of the formula

$$\frac{\delta\mathcal{H}}{\delta\mathbf{p}}=\operatorname{curl}\left(\frac{\delta\mathcal{H}}{\delta\mathbf{\Omega}}\right), \tag{16}$$

so the next equation for slow dynamics of the vorticity follows from Eq. (14):

$$\mathbf{\Omega}_t=\operatorname{curl}\left[\operatorname{curl}\left(\frac{\delta\mathcal{H}}{\delta\mathbf{\Omega}}\right)\times\frac{\mathbf{\Omega}}{\rho_0(\mathbf{r})}\right]. \tag{17}$$

This equation differs only by the presence of the function $\rho_0(\mathbf{r})$ (instead of the unity) from the equation used in [10] and [12] as a starting point in the transition to the vortex line representation in homogeneous systems. Therefore, all further constructions will be done similarly to Ref. [12]. First, let us fix the topology of the vorticity field by means of the formula

$$\mathbf{\Omega}(\mathbf{r},t)=\int\delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t)d\mathbf{a}$$
$$=\left.\frac{(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t)}{\det\|\partial\mathbf{R}/\partial\mathbf{a}\|}\right|_{\mathbf{a}=\mathbf{R}^{-1}(\mathbf{r},t)}, \tag{18}$$

where $\mathbf{\Omega}_0(\mathbf{a})$ is the Cauchy invariant. The vector

$$\mathbf{T}(\mathbf{a},t)=(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t) \tag{19}$$

is directed along the vorticity field at the point $\mathbf{r}=\mathbf{R}(\mathbf{a},t)$. It is necessary to stress that the information supplied by the mapping $\mathbf{R}(\mathbf{a},t)$ is not so full as the information supplied by the purely Lagrangian mapping $\mathbf{x}(\mathbf{a},t)$. The role of the mapping $\mathbf{R}(\mathbf{a},t)$ is exhausted by a continuous deformation of the vortex lines of the initial field $\mathbf{\Omega}_0$. This means that the Jacobian

$$J=\det\|\partial\mathbf{R}/\partial\mathbf{a}\| \tag{20}$$

is not related directly to the density $\rho_0(\mathbf{r})$, inasmuch as, unlike the mapping $\mathbf{x}(\mathbf{a},t)$, the new mapping $\mathbf{R}(\mathbf{a},t)$ is defined up to an arbitrary nonuniform shift along the vortex lines. Geometrical meaning of representation (18) becomes clearer if instead of $\mathbf{a}$ we use a so-called vortex line coordinate system $[\nu_1(\mathbf{a}),\nu_2(\mathbf{a}),\xi(\mathbf{a})]$, so that the 2D Lagrangian coordinate $\nu=(\nu_1,\nu_2)\in\mathcal{N}$ is a label of vortex lines, which lies in some manifold $\mathcal{N}$, while a longitudinal coordinate ξ parametrizes the vortex line. Locally, vortex line coordinate system exists for arbitrary topology of the vorticity field, but globally–only in the case when all the lines are closed. In the last case Eq. (18) can be rewritten in the simple form

$$\mathbf{\Omega}(\mathbf{r},t)=\int_{\mathcal{N}}d^2\nu\oint\delta(\mathbf{r}-\mathbf{R}(\nu,\xi,t))\mathbf{R}_\xi\,d\xi, \tag{21}$$

where $\mathbf{R}_\xi=\partial\mathbf{R}/\partial\xi$. The geometrical meaning of this formula is rather evident—the frozen-in vorticity field is presented as a continuous distribution of vortex lines. It is also clear that the choice of the longitudinal parameter is nonunique. This choice is determined exclusively by convenience for a particular task. Usage of the formula

$$\mathbf{\Omega}_t(\mathbf{r},t)=\operatorname{curl}_{\mathbf{r}}\int\delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))[\mathbf{R}_t(\mathbf{a},t)\times\mathbf{T}(\mathbf{a},t)]d\mathbf{a}, \tag{22}$$

which follows immediately from Eq. (18), together with the general relationship between variational derivatives of an arbitrary functional $F\{\mathbf{\Omega}\}$,

$$\left[\mathbf{T}\times\operatorname{curl}_{\mathbf{r}}\left(\frac{\delta F}{\delta\mathbf{\Omega}(\mathbf{R})}\right)\right]=\left.\frac{\delta F\{\mathbf{\Omega}\{\mathbf{R}\}\}}{\delta\mathbf{R}(\mathbf{a})}\right|_{\mathbf{\Omega}_0}, \tag{23}$$

allows us to obtain the equation of motion for the mapping $\mathbf{R}(\mathbf{a},t)$ by substitution of representation (18) into Eq. (17). As the result, dynamics of the mapping $\mathbf{R}(\mathbf{a},t)$ is determined by the equation

$$[(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a})\times\mathbf{R}_t(\mathbf{a})]\rho_0(\mathbf{R})=\frac{\delta\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R}\}\}}{\delta\mathbf{R}(\mathbf{a})}. \tag{24}$$

It is not very difficult to check by a direct calculation that the given equation of motion for $\mathbf{R}(\mathbf{a},t)$ follows from the variational principle $\delta\int\mathcal{L}_{\mathbf{\Omega}_0}dt=0$, where the Lagrangian is

$$\mathcal{L}_{\mathbf{\Omega}_0}=\int([\mathbf{R}_t\times\mathbf{D}(\mathbf{R})]\cdot(\mathbf{\Omega}_0\nabla_{\mathbf{a}})\mathbf{R})d\mathbf{a}-\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R}\}\}, \tag{25}$$

with the vector function $\mathbf{D}(\mathbf{R})$ being related to the density $\rho_0(\mathbf{r})$ by the equality

$$(\nabla_{\mathbf{R}}\cdot\mathbf{D}(\mathbf{R}))=\rho_0(\mathbf{R}). \tag{26}$$

For application to vortex filaments, the following form of the Lagrangian is more useful, where $\mathbf{R}=\mathbf{R}(\nu,\xi,t)$:

$$\mathcal{L}_{\mathcal{N}}=\int_{\mathcal{N}} d^2\nu \oint ([\mathbf{R}_t\times\mathbf{D}(\mathbf{R})]\cdot\mathbf{R}_\xi)d\xi-\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R}\}\}. \quad (27)$$

It should be stressed that conservation in time of the fluid amount inside each closed frozen-in vortex surface is not imposed *a priori* as a constraint for the mapping $\mathbf{R}(\mathbf{a},t)$. All such quantities are conserved in the dynamical sense due to the symmetry of the Lagrangian (27) with respect to the group of relabelings of the labels ν of vortex lines

$$\nu=\nu(\tilde{\nu},t), \qquad \partial(\nu_1,\nu_2)/\partial(\tilde{\nu}_1,\tilde{\nu}_2)=1. \quad (28)$$

Considering all one-parametrical subgroups of the given group of area-preserving transformations and applying Noether's theorem [2] to the Lagrangian (27), it is possible to obtain the indicated integrals of motion in the following form (compare with Ref. [21]):

$$I_\Psi=\int_{\mathcal{N}}\Psi(\nu_1,\nu_2)d^2\nu \oint \rho_0(\mathbf{R})([\mathbf{R}_1\times\mathbf{R}_2]\cdot\mathbf{R}_\xi)d\xi, \quad (29)$$

where $\Psi(\nu_1,\nu_2)$ is an arbitrary function on the manifold $\mathcal{N}$ of labels, with the only condition $\Psi|_{\partial\mathcal{N}}=0$.

IV. LOCAL INDUCTION APPROXIMATION

When a particular task is being solved, the necessity always arises in making some simplifications. The variational formulation for the dynamics of vortex lines allows us to introduce and control various approximations on the level of the Lagrangian (27), what in practice is more convenient and more simple than control of approximations made on the level of equations of motion. For example we will consider now the so called LIA in dynamics of a slender nonstretched vortex filament. As known, in spatially homogeneous systems the LIA yields an integrable equation that is gauge equivalent to the nonlinear Schrödinger equation [22]. In general case, inhomogeneity destroys the integrability of LIA equation. Nevertheless, this does not reduce the value of LIA as a simplified model of filament dynamics.

A. LIA in Eulerian hydrodynamics

At first, we will consider the Eulerian hydrodynamics, where the canonical momentum and the velocity coincide. Let the vorticity be concentrated in a quasi-one-dimensional structure, a vortex filament, with a typical longitudinal scale L being much larger than the width d of the filament. A typical scale of spatial inhomogeneity is supposed to be of order of L or larger. In such situation, the kinetic energy of the fluid is concentrated in the vicinity of the filament, with the corresponding integral being logarithmically large on the parameter L/d. The LIA consist in the following simplifications. First, in the kinetic part of the Lagrangian (27), the dependence of the shape of vortex lines on the label ν is neglected, i.e., the filament is considered as a single curve $\mathbf{R}(\xi,t)$. After integration over $d^2\nu$ the constant multiplier Γ appears now, which is the value of velocity circulation around the filament. Second, some significant simplifications may be done in the Hamiltonian. Generally speaking, exact expression for the Hamiltonian implies derivation of the dependence $\mathbf{v}\{\rho_0,\mathbf{\Omega}\}$ from the following system of equations:

$$\mathrm{curl}\,\mathbf{v}=\mathbf{\Omega}, \qquad \mathrm{div}(\rho_0(\mathbf{r})\cdot\mathbf{v})=0,$$

and subsequent substitution of $\mathbf{v}$ into the expression for the kinetic energy. After that one has to deal with a nonlocal Hamiltonian

$$\mathcal{H}^{\{\rho_0\}}_{Euler}=\frac{1}{2}\int\int G^{\{E,\rho_0\}}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\Omega_\alpha(\mathbf{r}_1)\Omega_\beta(\mathbf{r}_2)d\mathbf{r}_1\,d\mathbf{r}_2,$$

where the Green function $G^{\{E,\rho_0\}}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$ has the following asymptotics at close arguments:

$$G^{\{E,\rho_0\}}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\to\frac{\rho_0(\mathbf{r}_1)\,\delta_{\alpha\beta}}{4\pi|\mathbf{r}_2-\mathbf{r}_1|}, \qquad \mathbf{r}_2\to\mathbf{r}_1.$$

Therefore, the Hamiltonian of a singular vortex filament,

$$\mathcal{H}^{d=0}_f=\frac{\Gamma^2}{2}\oint\oint G^{\{E,\rho_0\}}_{\alpha\beta}(\mathbf{R}_1,\mathbf{R}_2)R'_{1\alpha}R'_{2\beta}\,d\xi_1\,d\xi_2, \quad (30)$$

where $R'_{1\alpha}=\partial_{\xi_1}R_\alpha(\xi_1)$ and so on, logarithmically diverges. Taking into account the finite width d and the longitudinal scale L, it is possible to put, with a logarithmic accuracy, the Hamiltonian of a thin vortex filament equal to the following expression:

$$\mathcal{H}^d_f\approx\mathcal{H}_A=\Gamma A\oint\rho_0(\mathbf{R})|\mathbf{R}_\xi|d\xi, \quad (31)$$

where the constant A is

$$A=\frac{\Gamma}{4\pi}\ln\left(\frac{L}{d}\right). \quad (32)$$

In accordance with the simplifications made above, the motion of a slender vortex filament in the spatially inhomogeneous system is described approximately by the equation

$$[\mathbf{R}_\xi\times\mathbf{R}_t]\rho_0(\mathbf{R})/A=\nabla\rho_0(\mathbf{R})\cdot|\mathbf{R}_\xi|-\partial_\xi\left(\rho_0(\mathbf{R})\cdot\frac{\mathbf{R}_\xi}{|\mathbf{R}_\xi|}\right),$$

which is obtained by substitution of the Hamiltonian (31) into Eq. (24). The given equation can be solved with respect to $\mathbf{R}_t$ and rewritten in terms of the geometrically invariant objects $\mathbf{t},\mathbf{b},\kappa$, where $\mathbf{t}$ is the unit tangent vector on the curve, $\mathbf{b}$ is the unit binormal vector, and κ is the curvature of the line. As the result, we have the equation

$$\mathbf{R}_t/A=[\boldsymbol{\nabla}\{\ln\rho_0(\mathbf{R})\}\times\mathbf{t}]+\kappa\mathbf{b}, \quad (33)$$

the applicability of which is not actually limited by the Eulerian hydrodynamics. Let us indicate at least two more physical models where the LIA equation (33) is useful.

B. LIA in EMHD

The first model is EMHD, the Hamiltonian of which contains, besides the kinetic energy, also the energy of magnetic field $\mathbf{B}$ created by current of the electron fluid through the motionless inhomogeneous ion fluid. In principle, the Hamiltonian of EMHD is determined by the relations that follow from the Lagrangian $\mathcal{L}_e$, Eq. (3):

$$\mathrm{curl}\,\mathbf{v}+\frac{e}{mc}\mathbf{B}=\mathbf{\Omega},\qquad \mathrm{curl}\,\mathbf{B}=\frac{4\pi e}{mc}\rho_0(\mathbf{r})\cdot\mathbf{v},$$

$$\mathcal{H}_{EMHD}=\int\left(\rho_0(\mathbf{r})\frac{\mathbf{v}^2}{2}+\frac{\mathbf{B}^2}{8\pi}\right)d\mathbf{r}.$$

In a spatially homogeneous system we would obtain the expression

$$\mathcal{H}^h_{EMHD}=\frac{\rho_0}{8\pi}\int\int\frac{e^{-q|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}\mathbf{\Omega}(\mathbf{r}_1)\cdot\mathbf{\Omega}(\mathbf{r}_2)d\mathbf{r}_1\,d\mathbf{r}_2,$$

where the screening parameter q is determined by the relation

$$q^2=\frac{4\pi\rho_0 e^2}{m^2c^2}.$$

In an inhomogeneous system q is a function of coordinates, with a typical value $\tilde{q}$. Let us suppose the inequalities $\tilde{q}L\gg 1$ and $\tilde{q}d\ll 1$. One can see that the logarithmic integral analogous to the expression (30) is cut now not on the L, but on the skin depth $\lambda=1/q$. Accordingly, for this case the constant A in LIA equation (33) is given by the expression

$$A_{EMHD}=(\Gamma/4\pi)\ln\left(\frac{mc}{ed\sqrt{\bar{\rho}}}\right).$$

We see that in ideal EMHD the LIA works better than in Eulerian hydrodynamics, due to the screening effect.

C. LIA in Bose-Einstein condensates

Another important physical model, where the equation (33) may be applied, is the theory of Bose-Einstein condensate for a weakly nonideal trapped gas with a quantized vortex filament [23]. At zero temperature this system is described approximately by the complex order parameter $\Phi(\mathbf{r},t)$ (the wave function of the condensate), with the equation of motion (the Gross-Pitaevskii equation) taking in dimensionless variables the form

$$i\Phi_t=\left(-\frac{1}{2}\Delta+U(\mathbf{r})-\mu+|\Phi|^2\right)\Phi,\qquad (34)$$

where $U(\mathbf{r})$ is an external potential, usually of the quadratic form

$$U(\mathbf{r})=ax^2+by^2+cz^2,$$

and the constant μ is the chemical potential. Let us suppose $a\geqslant b\geqslant c$. It is well known that Eq. (34) admits the hydrodynamical interpretation. The variables ρ and $\mathbf{p}$ are defined by the relations

$$\rho=|\Phi|^2,\qquad \rho\mathbf{p}=(\bar{\Phi}\nabla\Phi-\Phi\nabla\bar{\Phi})/2i.$$

The corresponding Hamiltonian is

$$\mathcal{H}_{GP}=\int\left[\frac{(\nabla\sqrt{\rho})^2+\rho\mathbf{p}^2)}{2}+[U(\mathbf{r})-\mu]\rho+\frac{\rho^2}{2}\right]d\mathbf{r}.$$

In comparison with the ordinary Eulerian hydrodynamics, there is the term depending on the density gradient in this expression. However, with large values of the parameter μ^2/a, one may neglect that term in the calculation of the equilibrium density inside the space region where the density is not exponentially small, and use the approximate formula

$$\rho_0(\mathbf{r})\approx\mu-U(\mathbf{r}),\qquad \text{if}\quad \{\mu^2\gg a,\mu-U(\mathbf{r})>0\}.$$

As is known, Eq. (34) admits solutions with quantized vortex filaments, the circulation around them being equal to 2π. In these solutions, the density differs significantly from $\rho_0(\mathbf{r})$ only at close distances of order $1/\sqrt{\mu}$ from the zero line. Far away, up to distances of order $L\sim\sqrt{\mu/a}\gg 1/\sqrt{\mu}$, we have almost Eulerian flow. Therefore, the LIA equation (33) is valid for a description of slow motion of the quantum vortex filament in trapped Bose-condensates of a relatively large size L, with the parameter

$$A=A_{GP}=(1/4)\ln(\mu^2/a).$$

The inequality $\mu^2\gg a$ ensures also the smallness of the filament velocity $v_f\sim\kappa A_{GP}$ with respect to the speed of sound $c_s\sim\sqrt{\mu}$, while the curvature of the filament is of order $\kappa\sim\sqrt{a/\mu}$.

V. CONCLUSIONS

Let us summarize briefly the main results of this paper. First, neglecting acoustic degrees of freedom in the investigation of slow isentropic flows of a compressible perfect fluid in spatially inhomogeneous systems, the general form of variational principle for the dynamics of frozen-in vortex lines has been found. The connection from the basic Lagrangian given in terms of density and velocity fields to the Hamiltonian of vortex lines has been provided, which allows one to analyze vorticity dynamics in complicated systems initially specified by the principle of least action. Second, this method has been applied to several physically important models, such as Eulerian hydrodynamics in an external field, ideal electron magnetohydrodynamics on inhomogeneous ion background, and the Gross-Pitaevskii model for trapped Bose-Einstein condensate, in order to derive approximate equations of motion for vortex filaments. It has been established that a mathematical structure of the equations derived is the same in each of these three cases, though the parameters have different physical meaning in each case.

The final remark concerns the possibility of development of an analogous approach in the general case, where acoustic waves are important. Although at present this work has not been completed, the author hopes it will be done in the future on the basis of Eqs. (13) and (14).

ACKNOWLEDGMENTS

This work was supported by RFBR (Grant No. 00-01-00929), by the Russian State Program of Support of the Leading Scientific Schools (Grant No. 00-15-96007), by the INTAS, and by the Fund KFA, Forschungszentrum, Jülich, Germany.

[1] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer-Verlag, New York, 1989).
[2] B.A. Dubrovin, S.P. Novikov, and A.T. Fomenko, *Modern Geometry* (Nauka, Moscow, 1979).
[3] V.E. Zakharov and E.A. Kuznetsov, Usp. Fiz. Nauk **167**, 1137 (1997) [Phys. Usp. **40**, 1087 (1997)].
[4] P.J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).
[5] R. Salmon, Annu. Rev. Fluid Mech. **20**, 225 (1988).
[6] N. Padhye and P.J. Morrison, Plasma Phys. Rep. **22**, 869 (1996).
[7] V.I. Il'gisonis and V.P. Lakhin, Plasma Phys. Rep. **25**, 58 (1999).
[8] V. Berdichevsky, in *Thermodynamics of Chaos and Order* (Addison-Wesley, Reading, MA, 1997), p. 227.
[9] V. Berdichevsky, Phys. Rev. E **57**, 2885 (1998).
[10] E.A. Kuznetsov and V.P. Ruban, Pis'ma Zh. Eksp. Teor. Fiz. **67**, 1015 (1998) [JETP Lett. **67**, 1076 (1998)].
[11] V.P. Ruban, Zh. Eksp. Teor. Fiz. **116**, 563 (1999) [JETP **89**, 299 (1999)].
[12] E.A. Kuznetsov and V.P. Ruban, Phys. Rev. E **61**, 831 (2000).
[13] H. Lamb, *Hydrodynamics*, 6th ed. (Cambridge University Press, Cambridge, 1932) [Russian translation, Gostekhizdat, Moscow, 1947].
[14] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1987).
[15] M.I. Monastyrskii and P.V. Sasorov, Zh. Eksp. Teor. Fiz. **93**, 1210 (1987) [Sov. Phys. JETP **66**, 683 (1987)].
[16] A. Ruzmaikin and P. Akhmetiev, Phys. Plasmas **1**, 331 (1994).
[17] V.P. Ruban, D.I. Podolsky, and J.J. Rasmussen, Phys. Rev. E **63**, 056306 (2001).
[18] V.P. Ruban, Phys. Rev. D **62**, 127504 (2000).
[19] V.P. Ruban and D.I. Podolsky, e-print gr-qc/0102087; Phys. Rev. D (to be published).
[20] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1980).
[21] V.P. Ruban, Phys. Rev. E **62**, 4950 (2000).
[22] H. Hasimoto, J. Fluid Mech. **51**, 477 (1972).
[23] F. Dalfovo *et al.*, Rev. Mod. Phys. **71**, 463 (1999).